ON LOCAL TORUS ACTIONS MODELED ON THE STANDARD REPRESENTATION(The theory of transformation groups and its applications)

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1. INTRODUCTION

Let S^1 be the unit circle and $T^n := (S^1)^n$ the n-dimensional compact torus. T^n acts on the n-dimensional complex vector space $\mathbb{C}^n$ by coordinate-wise complex multiplication. This action is called the *standard representation of* T^n. T^n acts on a complex n-dimensional toric variety X as a subgroup of $(\mathbb{C}^n)^*$. If X is nonsingular, then for each point $x \in X$, there exists a coordinate neighborhood (U, ρ, φ) of x, where where U is a T^n-invariant open set of X, ρ is an automorphism of T^n, and φ is a ρ-equivariant diffeomorphism from U to some T^n-invariant open subset in $\mathbb{C}^n$. In general, a T^n-action on a $2n$-dimensional manifold which is covered by such coordinate neighborhoods is said to be *locally standard*. See [4, 2] for more details. This property is one of the starting point of their pioneer work [4] of Davis-Januszkiewicz and now, it plays a central role in toric topology.

A similar structure can be seen in Lagrangian fibrations. Let (X, ω) be a $2n$-dimensional smooth symplectic manifold and B an n-dimensional smooth manifold with corners. We call a map $\mu\colon (X, \omega) \to B$ a *locally toric Lagrangian fibration* if μ is locally identified with the moment map of the standard representation of T^n. It is known that there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of X and there also exists an automorphism ρ of T^n on each nonempty overlap $U_\alpha \cap U_\beta$ such that each φ_α sends U_α diffeomorphically to some T^n-invariant open subset of $\mathbb{C}^n$ and the overlap map $\varphi_\alpha^X \circ (\varphi_\beta^X)^{-1}$ is ρ-equivariant (see also Example 2.9).

In [13], as a generalization of a locally standard torus action and also as an underlying structure of a locally toric Lagrangian fibration, we introduced the notion of a local T^n-action modeled on the standard representation, and defined two topological invariants called the characteristic pair and the Euler class of the orbit map for a local T^n-action, then proved that local T^n-actions are topologically classified by these two invariants. We also investigate the symplectic case. The content of [13] is a refinement of the work [12].

This is an announcement of [13]. In the next section, we recall the definition and basic facts of a local T^n-action. In Section 3, we explain that a local T^n-action is accompanied by the principal $\mathrm{Aut}(T^n)$-bundle and the characteristic bundle. After that, we recall the construction of the canonical model of a local T^n-action. In Section 4, we define the Euler class of the orbit map. Section 5 is devoted to the topological classification of local T^n-actions. Theorem 5.1 is the main theorem of the first part of this paper. We also describe the idea of the proof, where the canonical model plays an important role. As a corollary, we can obtain that locally standard T^n-actions are classified by the characteristic bundle and the Euler class of the orbit map up to equivariant homeomorphisms (Corollary 5.2). One of the important examples of manifolds equipped with local T^n-actions is a locally toric Lagrangian fibration with n-dimensional fibers. Finally, in Section 6, we give the

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necessary and sufficient condition that a manifold with a local T^n-action becomes a locally toric Lagrangian fibration and also describe the classification of locally toric Lagrangian fibrations.

Throughout this paper we employ the vector notation in order to represent elements of $\mathbb{C}^n$, namely, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. The similar notation is also used for $T^n = (S^1)^n$, $\mathbb{R}^n$, etc.

2. Definitions and basic facts

Let X be a paracompact, Hausdorff space.

Definition 2.1. A *weakly standard* C^r $(0 \leq r \leq \infty)$ *atlas* of X is an atlas $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}}$ which satisfies the following properties

1. for each α, φ_α^X is a homeomorphism from U_α^X to an open set of $\mathbb{C}^n$ invariant under the standard representation of T^n and
2. for each nonempty overlap $U_{\alpha\beta}^X := U_\alpha^X \cap U_\beta^X$, there exists an automorphism $\rho_{\alpha\beta}$ of T^n as a Lie group such that the overlap map $\varphi_{\alpha\beta}^X := \varphi_\alpha^X \circ (\varphi_\beta^X)^{-1}$ is $\rho_{\alpha\beta}$-equivariant C^r diffeomorphic with respect to the restrictions of the standard representation of T^n to $\varphi_\alpha^X(U_{\alpha\beta}^X)$ and $\varphi_\beta^X(U_{\alpha\beta}^X)$. (The latter means that $\varphi_{\alpha\beta}^X(u \cdot z) = \rho_{\alpha\beta}(u) \cdot \varphi_{\alpha\beta}^X(z)$ for $u \in T^n$ and $z \in \varphi_\beta^X(U_{\alpha\beta}^X)$.)

Two weakly standard C^r atlases $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}}$ and $\{(V_\beta^X, \psi_\beta^X)\}_{\beta \in \mathcal{B}}$ of X^{2n} are *equivalent* if on each nonempty overlap $U_\alpha^X \cap V_\beta^X$, there exists an automorphism ρ of T^n such that $\varphi_\alpha^X \circ (\psi_\beta^X)^{-1}$ is ρ-equivariant C^r diffeomorphic. We call an equivalence class of weakly standard C^r atlases a C^r *local* T^n*-action on* X^{2n} *modeled on the standard representation* and denote it by $\mathcal{T}$.

In the rest of this paper, a C^r local T^n-action on X^{2n} modeled on the standard representation is often called a C^r local T^n-action on X^{2n}, or more simply, a local T^n-action on X if there are no confusions.

Let $(X, \mathcal{T})$ be a $2n$-dimensional manifold X equipped with a C^r local T^n-action $\mathcal{T}$ and $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}}$ a maximal weakly standard atlas of X which belongs to $\mathcal{T}$. For $(X, \mathcal{T})$ we can generalize the orbit space and the orbit map in the following way. We endow each quotient space $\varphi_\alpha^X(U_\alpha^X)/T^n$ with the quotient topology induced from the topology of $\varphi_\alpha^X(U_\alpha^X)$ by the natural projection $\pi\colon \varphi_\alpha^X(U_\alpha^X) \to \varphi_\alpha^X(U_\alpha^X)/T^n$. By the property (2) for each overlap $U_{\alpha\beta}^X$, $\varphi_{\alpha\beta}^X$ induces a homeomorphism from $\varphi_\beta^X(U_{\alpha\beta}^X)/T^n$ to $\varphi_\alpha^X(U_{\alpha\beta}^X)/T^n$. We define two elements $b_\alpha \in \varphi_\alpha^X(U_\alpha^X)/T^n$ and $b_\beta \in \varphi_\beta^X(U_\beta^X)/T^n$ are equivalent if $b_\alpha \in \varphi_\alpha^X(U_{\alpha\beta}^X)/T^n$, $b_\beta \in \varphi_\beta^X(U_{\alpha\beta}^X)/T^n$ and the map induced by $\varphi_{\alpha\beta}^X$ sends b_β to b_α. It is an equivalence relation on the disjoint union $\coprod_\alpha \left(\varphi_\alpha^X(U_\alpha^X)/T^n\right)$. We call the quotient space of $\coprod_\alpha \left(\varphi_\alpha^X(U_\alpha^X)/T^n\right)$ by the equivalence relation together with a quotient topology the *orbit space* of the local T^n-action $\mathcal{T}$ on X and denote it by B_X. It is easy to see that B_X is a Hausdorff space and $\{\varphi_\alpha^X(U_\alpha^X)/T^n\}_{\alpha \in \mathcal{A}}$ is an open covering of B_X. By the construction of B_X, the map $\coprod_\alpha \pi \circ \varphi_\alpha^X\colon \coprod_\alpha U_\alpha^X \to \coprod_\alpha \left(\varphi_\alpha^X(U_\alpha^X)/T^n\right)$ induces the map from X to B_X. We call it the *orbit map* of the local T^n-action $\mathcal{T}$ on X and denote it by $\mu_X\colon X \to B_X$. Notice that by the construction, it is a continuous open map.

Let $\mathbb{R}_+^n$ be the standard n-dimensional positive cone

$$\mathbb{R}_+^n := \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \colon \xi_i \geq 0 \ i = 1, \dots, n\}.$$

It has the natural stratification with respect to the number of coordinates ξ_i which are equal to zero.

Definition 2.2. Let B be a Hausdorff space. A *structure of an n-dimensional topological manifold with corners* on B is a system of coordinate neighborhoods onto open subsets of $\mathbb{R}^n_+$ so that overlap maps are homeomorphisms which preserve the natural stratifications induced from the one of $\mathbb{R}^n_+$. See [3, Section 6] for a topological manifold with corners.

Proposition 2.3. *B_X is endowed with a structure of an n-dimensional topological manifold with corners.*

Proof. We define the map $\mu_{\mathbb{C}^n}\colon \mathbb{C}^n \to \mathbb{R}^n$ by

$$\mu_{\mathbb{C}^n}(z) = (|z_1|^2, \dots, |z_n|^2) \tag{2.1}$$

for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Notice that the image of $\mu_{\mathbb{C}^n}$ is the n-dimensional standard positive cone $\mathbb{R}^n_+$. It is invariant under the standard representation of T^n and induces the homeomorphism from $\mathbb{C}^n/T^n$ to $\mathbb{R}^n_+$. The orbit space $\mathbb{C}^n/T^n$ is endowed with the natural stratification whose k-dimensional stratum consists of k-dimensional orbits and the homeomorphism induced by $\mu_{\mathbb{C}^n}$ preserves stratifications of $\mathbb{C}^n/T^n$ and $\mathbb{R}^n_+$. We put $U^B_\alpha := \varphi^X_\alpha(U^X_\alpha)/T^n$. The restriction of $\mu_{\mathbb{C}^n}$ to $\varphi^X_\alpha(U^X_\alpha)$ induces the homeomorphism from U^B_α to the open subset $\mu_{\mathbb{C}^n}(\varphi^X_\alpha(U^X_\alpha))$ of $\mathbb{R}^n_+$, which is denoted by φ^B_α. By the construction, on each overlap $U^B_{\alpha\beta} := U^B_\alpha \cap U^B_\beta$, the overlap map $\varphi^B_{\alpha\beta} := \varphi^B_\alpha \circ (\varphi^B_\beta)^{-1}\colon \mu_{\mathbb{C}^n}(\varphi^X_\beta(U^X_{\alpha\beta})) \to \mu_{\mathbb{C}^n}(\varphi^X_\alpha(U^X_{\alpha\beta}))$ preserves the natural stratifications of $\mu_{\mathbb{C}^n}(\varphi^X_\alpha(U^X_{\alpha\beta}))$ and $\mu_{\mathbb{C}^n}(\varphi^X_\beta(U^X_{\alpha\beta}))$. Thus, $\{(U^B_\alpha, \varphi^B_\alpha)\}_{\alpha\in\mathcal{A}}$ is the desired atlas. □

Remark 2.4. The atlas $\{(U^B_\alpha, \varphi^B_\alpha)\}_{\alpha\in\mathcal{A}}$ of B_X constructed in the proof of Proposition 2.3 has following properties

(1) for each α, $U^X_\alpha = \mu_X^{-1}(U^B_\alpha)$, $\varphi^X_\alpha(U^X_\alpha) = \mu^{-1}_{\mathbb{C}^n}(\varphi^B_\alpha(U^B_\alpha))$ and the following diagram commutes

$$\begin{array}{ccccccc}
X & \supset & \mu_X^{-1}(U^B_\alpha) & \xrightarrow{\varphi^X_\alpha} & \mu^{-1}_{\mathbb{C}^n}(\varphi^B_\alpha(U^B_\alpha)) & \subset & \mathbb{C}^n \\
\downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_{\mathbb{C}^n}} & & \downarrow{\scriptstyle \mu_{\mathbb{C}^n}} \\
B_X & \supset & U^B_\alpha & \xrightarrow{\varphi^B_\alpha} & \varphi^B_\alpha(U^B_\alpha) & \subset & \mathbb{R}^n_+,
\end{array}$$

(2) the restriction of $\{(U^B_\alpha, \varphi^B_\alpha)\}_{\alpha\in\mathcal{A}}$ to the interior $B_X \setminus \partial B_X$ of B_X is a C^r atlas of $B_X \setminus \partial B_X$.

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be $2n$-dimensional manifolds X_1 and X_2 equipped with C^r local T^n-actions $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $\{(U^{X_1}_\alpha, \varphi^{X_1}_\alpha)\}_{\alpha\in\mathcal{A}}$ and $\{(U^{X_2}_\beta, \varphi^{X_2}_\beta)\}_{\beta\in\mathcal{B}}$ be the maximal weakly standard atlases of X_1 and X_2 which belong to $\mathcal{T}_1$ and $\mathcal{T}_2$.

Definition 2.5. $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are C^r *isomorphic* if there exists a C^r diffeomorphism $f_X\colon X_1 \to X_2$ from X_1 to X_2 and on each nonempty overlap $U^{X_1}_\alpha \cap (f_X)^{-1}(U^{X_2}_\beta) \neq \emptyset$ there exists an automorphism ρ of T^n such that $\varphi^{X_2}_\beta \circ f_X \circ (\varphi^{X_1}_\alpha)^{-1}$ is ρ-equivariant. We also call such a C^r diffeomorphism f_X a C^r *isomorphism* and denote it by $f_X\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$.

Notice that a C^r isomorphism $f_X\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ induces the stratification preserving homeomorphism $f_B\colon B_{X_1} \to B_{X_2}$ between their orbit spaces such that f_X and f_B satisfy $\mu_{X_2} \circ f_X = f_B \circ \mu_{X_1}$.

We give examples of local torus actions.

Example 2.6 (Locally standard torus actions). Let T^n act smoothly on a $2n$-dimensional smooth manifold X. A *standard coordinate neighborhood* of X consists of a triple (U, ρ, φ), where U is a T^n-invariant open set of X, ρ is an automorphism of T^n, and φ is a ρ-equivariant diffeomorphism from U to some T^n-invariant open subset in $\mathbb{C}^n$. The action of T^n on X is said to be *locally standard* if every point in X lies in some standard coordinate neighborhood. See [4, 2] for more details.

(A typical example of locally standard torus actions is a nonsingular toric variety.) The atlas which consists of standard coordinate neighborhoods is weakly standard. Therefore, a locally standard T^n-action induces the local T^n-action on X.

Notice that not all local torus actions are induced by locally standard torus actions. For any C^r local T^n-action $\mathcal{T}$ on a $2n$-dimensional manifold X, we take a weakly standard atlas $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$ belonging to $\mathcal{T}$. It is easy to see that the automorphisms $\rho_{\alpha\beta}$ of T^n in the property (2) of Definition 2.1 form a Čech one-cocycle $\{\rho_{\alpha\beta}\}$ on $\{U_\alpha^B\}_{\alpha\in\mathcal{A}}$ with values in $\mathrm{Aut}(T^n)$. Then, the cohomology class of $\{\rho_{\beta\alpha}\}$ in the first Čech cohomology set $H^1(B_X; \mathrm{Aut}(T^n))$ is the obstruction for the local T^n-action to be induced by a locally standard T^n-action.

Proposition 2.7. *A C^r local T^n-action on X is induced by some C^r locally standard T^n-action if and only if $\{\rho_{\alpha\beta}\}$ and the trivial Čech one-cocycle are of the same equivalence class in $H^1(B_X; \mathrm{Aut}(T^n))$, where the trivial Čech one-cocycle is the one whose values on all open set are equal to the identity map of T^n.*

For the proof, see [13].

Example 2.8. We can construct an example of local torus actions which does not come from any locally standard torus fibrations in the following way. For a small positive number $0 < \varepsilon \ll 1$, let $\overline{X}$ be the quotient space of the space

$$\{(z,w) \in \mathbb{C}^2 \times \mathbb{C} \colon 0 < |z_1|^2 < 1+\varepsilon, |w|^2 + |z_2|^2 = 1\}$$

by the S^1-action defined by

$$u \cdot (z,w) := \left((z_1, u^{-1}z_2), u^{-1}w\right).$$

T^2 acts on $\overline{X}$ by

$$u \cdot [z,w] := [u\cdot z, w].$$

The map $\mu_{\overline{X}} \colon \overline{X} \to \mathbb{R}^2$ defined by $\mu_{\overline{X}}([z,w]) := (|z_1|^2, |z_2|^2)$ is invariant under the T^2-action and induces the identification of the orbit space of the T^2-action with $(0, 1+\varepsilon) \times [0,1]$.

We define that two elements $\overline{x}_1$ and $\overline{x}_2$ in $\overline{X}$ are equivalent, or $\overline{x}_1 \sim_X \overline{x}_2$ if for a representative (z,w) of $\overline{x}_1$, $\left((\overline{z_1}/|z_1|\sqrt{|z_1|^2+1}, \overline{z_2}), \overline{w}\right)$ is a representative of $\overline{x}_2$. It does not depend on the choice of representatives of $\overline{x}_1$ and it is well-defined. We denote the quotient space $\overline{X}/\sim_X$ of the equivalence relation by X. By the construction, we can show that X is endowed with a local T^n-action. The orbit space B_X is the cylinder defined by

$$B_X := (0, 1+\varepsilon) \times [0,1]/\sim_B,$$

where $\xi \sim_B \eta$ if and only if $\eta_1 = \xi_1 + 1$ and $\eta_2 = \xi_2$, and $\mu_{\overline{X}}$ induces the orbit map $\mu_X \colon X \to B_X$.

Example 2.9 (Locally toric Lagrangian fibrations [7]). Let $\omega_{\mathbb{C}^n} := \frac{1}{2\pi\sqrt{-1}}\sum_{k=1}^n dz_k \wedge d\overline{z}_k$ be the standard symplectic structure on $\mathbb{C}^n$. The standard representation of T^n preserves $\omega_{\mathbb{C}^n}$ and the map $\mu_{\mathbb{C}^n} \colon \mathbb{C}^n \to \mathbb{R}^n$ defined by (2.1) is a moment map of the standard representation of T^n. Notice that the image of $\mu_{\mathbb{C}^n}$ is the n-dimensional standard positive cone $\mathbb{R}^n_+$. Let (X,ω) be a $2n$-dimensional symplectic manifold and B an n-dimensional manifold with corners. A map $\mu \colon (X,\omega) \to B$ is called a *locally toric Lagrangian fibration* if there exists a system $\{(U_\alpha, \varphi_\alpha^B)\}$ of coordinate neighborhoods of B into $\mathbb{R}^n_+$, and for each α there exists a symplectomorphism $\varphi_\alpha^X \colon (\mu^{-1}(U_\alpha), \omega) \to (\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha)), \omega_{\mathbb{C}^n})$ such that $\mu_{\mathbb{C}^n} \circ \varphi_\alpha^X = \varphi_\alpha^B \circ \mu$. We show in [13] that for a locally toric Lagrangian fibration $\mu \colon (X,\omega) \to B$ on an n-dimensional base B and an above atlas $\{(U_\alpha, \varphi_\alpha^B, \varphi_\alpha^X)\}$, on each nonempty overlap $U_\alpha \cap U_\beta$ there exists an automorphism $\rho_{\alpha\beta} \in \mathrm{Aut}(T^n)$ such that the overlap map

$\varphi_\alpha^X \circ (\varphi_\beta^X)^{-1}$ is ρ-equivariant. (Precisely, $\rho_{\alpha\beta}$ is a map from $U_\alpha \cap U_\beta \to \mathrm{Aut}(T^n)$. Since $\mathrm{Aut}(T^n)$ is discrete, $\rho_{\alpha\beta}$ is locally constant.) In particular, X is endowed with a smooth local T^n-action. In Section 6, we will describe the necessary and sufficient condition that a manifold with a local torus action becomes a locally toric Lagrangian fibration.

3. Characteristic pairs and canonical models

In this section, we introduce the characteristic pair for a local torus action, and construct the canonical model from the characteristic pair. Both of them play important roles of the topological classification of local torus actions. In this section, all manifolds, maps, and local T^n-actions are assumed to be of class C^0 unless otherwise stated.

3.1. **Characteristic pairs.** Let B be an n-dimensional topological manifold with corners. We assume that $\partial B \neq \emptyset$. By the definition of a manifold with corners, B is equipped with a natural stratification. We denote by $\mathcal{S}^{(k)}B$ the k-dimensional stratum of B, namely, $\mathcal{S}^{(k)}B$ consists of those points which have exactly k nonzero components in a local coordinate. In particular, the top-dimensional stratum $\mathcal{S}^{(n)}B$ is equal to the interior $B \setminus \partial B$ of B.

Let $\Lambda := \{t \in \mathfrak{t}\colon \exp t = 1\}$ be the lattice of integral elements in the Lie algebra $\mathfrak{t}$ of T^n. Since the differential of any automorphism of T^n at the unit element preserves Λ, by associating any automorphism of T^n with its differential at the unit element, there is the natural homomorphism from $\mathrm{Aut}(T^n)$ to $\mathrm{GL}(\Lambda)$. It is an isomorphism. In fact, it follows from the surjectivity of the exponential map of T^n and the equation $\varphi \circ \exp = \exp \circ d\varphi$ for any automorphism $\varphi \in \mathrm{Aut}(T^n)$. In the rest of this paper, we identify $\mathrm{Aut}(T^n)$ with $\mathrm{GL}(\Lambda)$ by this isomorphism. Let $\pi_P\colon P \to B$ be a principal $\mathrm{Aut}(T^n)$-bundle on B and $\pi_\Lambda\colon \Lambda_P \to B$ the associated Λ-bundle of P by the above isomorphism $\mathrm{Aut}(T^n) \cong \mathrm{GL}(\Lambda)$. Suppose that $\pi_\mathcal{L}\colon \mathcal{L} \to \mathcal{S}^{(n-1)}B$ is a rank one sub-bundle of the restriction $\pi_\Lambda|_{\mathcal{S}^{(n-1)}B}\colon \Lambda_P|_{\mathcal{S}^{(n-1)}B} \to \mathcal{S}^{(n-1)}B$ of $\pi_\Lambda\colon \Lambda_P \to B$ to $\mathcal{S}^{(n-1)}B$. For each k and any point $b \in \mathcal{S}^{(k)}B$, let U be an open neighborhood of b in B on which there exists a local trivialization $\varphi^\Lambda\colon \pi_\Lambda^{-1}(U) \to U \times \Lambda$ of Λ_P. By shrinking U if necessary, we can assume that the intersection $U \cap \mathcal{S}^{(n-1)}B$ of U with $\mathcal{S}^{(n-1)}B$ has exactly $n-k$ connected components, say, $(U \cap \mathcal{S}^{(n-1)}B)_1, \ldots, (U \cap \mathcal{S}^{(n-1)}B)_{n-k}$. Since Λ is discrete, for each $(U \cap \mathcal{S}^{(n-1)}B)_a$ there exists a rank one sub-lattice $L_a \subset \Lambda$ such that φ^E sends the preimage $\pi_\mathcal{L}^{-1}\left((U \cap \mathcal{S}^{(n-1)}B)_a\right)$ of $(U \cap \mathcal{S}^{(n-1)}B)_a$ by $\pi_\mathcal{L}$ fiber-wisely to $(U \cap \mathcal{S}^{(n-1)}B)_a \times L_a$.

Definition 3.1. $\pi_\mathcal{L}\colon \mathcal{L} \to \mathcal{S}^{(n-1)}B$ is said to be *unimodular* if for each k and any point $b \in \mathcal{S}^{(k)}B$, the sub-lattice $L_1 + \cdots + L_{n-k}$ generated by $L_1, \ldots, L_{n-k}$ is a rank $n-k$ direct summand of Λ. (In [4] such a sub-lattice is called an *$(n-k)$-dimensional unimodular subspace* of Λ.)

Notice that rank one sub-lattices $L_1, \ldots, L_{n-k}$ depend on the choice of a neighborhood U and a local trivialization φ^E. But Definition 3.1 does not depend on the choice of them because the condition for a sub-lattice to be unimodular is invariant by an automorphism of Λ.

Definition 3.2. Let $\pi_\mathcal{L}\colon \mathcal{L} \to \mathcal{S}^{(n-1)}B$ be a unimodular rank one sub-bundle of $\pi_\Lambda|_{\mathcal{S}^{(n-1)}B}\colon \Lambda_P|_{\mathcal{S}^{(n-1)}B} \to \mathcal{S}^{(n-1)}B$. Then the pair $(P, \mathcal{L})$ of the principal $\mathrm{Aut}(T^n)$-bundle $\pi_P\colon P \to B$ and $\pi_\mathcal{L}\colon \mathcal{L} \to \mathcal{S}^{(n-1)}B$ is called a *characteristic pair* and $\pi_\mathcal{L}\colon \mathcal{L} \to \mathcal{S}^{(n-1)}B$ is called a *characteristic bundle.*

Let $(X, \mathcal{T})$ be a $2n$-dimensional manifold equipped with a local T^n-action. We show that there is a characteristic pair associated with $(X, \mathcal{T})$. Let $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha \in \mathcal{A}} \in \mathcal{T}$ be the maximal weakly standard atlas. It induces the atlas $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha \in \mathcal{A}}$ of

B_X which satisfies the properties in Remark 2.4 and also determines a Čech one-cocycle $\{\rho_{\alpha\beta}\}$ on $\{U^B_\alpha\}_{\alpha\in\mathcal{A}}$ with coefficients in $\mathrm{Aut}(T^n)$. It defines the principal $\mathrm{Aut}(T^n)$-bundle $\pi_{P_X} : P_X \to B_X$ on B_X by setting

$$(3.1) \qquad P_X := \left(\coprod_\alpha U^B_\alpha \times \mathrm{Aut}(T^n)\right) / \sim_P,$$

where $(b_\alpha, h_\alpha) \in U^B_\alpha \times \mathrm{Aut}(T^n) \sim_{P_X} (b_\beta, h_\beta) \in U^B_\beta \times \mathrm{Aut}(T^n)$ if and only if $b_\alpha = b_\beta \in U^B_{\alpha\beta}$ and $h_\alpha = \rho_{\alpha\beta} \circ h_\beta$. The bundle projection π_{P_X} is defined by the obvious way. For each α, every point in $\pi^{-1}_{P_X}(U^B_\alpha)$ has a unique representative which lies in $U^B_\alpha \times \mathrm{Aut}(T^n)$. By associating a point in $\pi^{-1}_{P_X}(U^B_\alpha)$ with the unique representative, we define the local trivialization of P_X on U^B_α which is denoted by $\varphi^P_\alpha : \pi^{-1}_{P_X}(U^B_\alpha) \to U^B_\alpha \times \mathrm{Aut}(T^n)$. Let $\pi_{\Lambda_X} : \Lambda_X \to B_X$ be the Λ-bundle associated with P_X by the natural identification $\mathrm{Aut}(T^n) \cong \mathrm{GL}(\Lambda)$. The property (2) in Definition 2.1 determines a unique unimodular sub-bundle of the restriction $\pi_{\Lambda_X}|_{\mathcal{S}^{(n-1)}B_X} : \Lambda_X|_{\mathcal{S}^{(n-1)}B_X} \to \mathcal{S}^{(n-1)}B_X$ of $\pi_{\Lambda_X} : \Lambda_X \to B_X$ to the codimension one stratum $\mathcal{S}^{(n-1)}B_X$ in the following way. For each coordinate neighborhood $(U^B_\alpha, \varphi^B_\alpha)$ of B_X with $U^B_\alpha \cap \mathcal{S}^{(n-1)}B_X \neq \emptyset$, the preimage $\mu^{-1}_{\mathbf{C}^n}(\varphi^B_\alpha(U^B_\alpha \cap \mathcal{S}^{(n-1)}B_X))$ is equipped with the T^n-action which is the restriction of the standard representation of T^n. For simplicity, we assume that the intersection $U^B_\alpha \cap \mathcal{S}^{(n-1)}B_X$ is connected. (Otherwise, we may consider component-wise.) Then, all points of $\mu^{-1}_{\mathbf{C}^n}(\varphi^B_\alpha(U^B_\alpha \cap \mathcal{S}^{(n-1)}B_X))$ has the common one-dimensional stabilizer with respect to the T^n-action. We denote it by S^1_α and also denote the rank one sub-lattice of Λ spanned by the integral element which generates S^1_α by $\mathcal{L}_\alpha$. Suppose that $(U^B_\alpha, \varphi^B_\alpha)$ and $(U^B_\beta, \varphi^B_\beta)$ are coordinate neighborhoods satisfying the above conditions and the intersection $U^B_{\alpha\beta} \cap \mathcal{S}^{(n-1)}B_X$ is nonempty. Since the overlap map $\varphi^X_{\alpha\beta}$ is a $\rho_{\alpha\beta}$-equivariant homeomorphism, we can show that $\rho_{\alpha\beta}$ sends S^1_β isomorphically to S^1_α. Under the identification of $\rho_{\alpha\beta}$ with the automorphism of Λ induced by $\rho_{\alpha\beta}$, $\rho_{\alpha\beta}$ also sends $\mathcal{L}_\beta$ isomorphically to $\mathcal{L}_\alpha$. By the construction of $\pi_{\Lambda_X} : \Lambda_X \to B_X$, φ^P_α induces a local trivialization φ^Λ_α: $\pi^{-1}_{\Lambda_X}(U^B_\alpha) \to U^B_\alpha \times \Lambda$ of $\pi_{\Lambda_X} : \Lambda_X \to B_X$ on each U^B_α such that on an overlap $U^B_{\alpha\beta}$ the transition function with respect to φ^Λ_α and φ^Λ_β is $\rho_{\alpha\beta}$. We take a subsystem $\{(U^B_{\alpha_i}, \varphi^B_{\alpha_i})\}_{i\in\mathcal{I}}$ of $\{(U^B_\alpha, \varphi^B_\alpha)\}_{\alpha\in\mathcal{A}}$ which covers $\mathcal{S}^{(n-1)}B_X$ and define the rank one sub-bundle $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to \mathcal{S}^{(n-1)}B_X$ of $\pi_{\Lambda_X}|_{\mathcal{S}^{(n-1)}B_X} : \Lambda_X|_{\mathcal{S}^{(n-1)}B_X} \to \mathcal{S}^{(n-1)}B_X$ by setting

$$(3.2) \qquad \mathcal{L}_X := \left(\coprod_i U^B_{\alpha_i} \cap \mathcal{S}^{(n-1)}B_X \times \mathcal{L}_{\alpha_i}\right) / \sim_L,$$

where $(b_i, l_i) \in U^B_{\alpha_i} \cap \mathcal{S}^{(n-1)}B_X \times \mathcal{L}_{\alpha_i} \sim_L (b_j, l_j) \in U^B_{\alpha_j} \cap \mathcal{S}^{(n-1)}B_X \times \mathcal{L}_{\alpha_j}$ if and only if $b_i = b_j$ and $l_i = \rho_{\alpha_i\alpha_j}(l_j)$. By the construction, it is easy to see that $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to \mathcal{S}^{(n-1)}B_X$ is unimodular. As a summary, we have the following proposition.

Proposition 3.3. *Associated with a local T^n-action $\mathcal{T}$ on X, there exists a characteristic pair $(P_X, \mathcal{L}_X)$, where P_X and $\mathcal{L}_X$ are defined by* (3.1) *and* (3.2), *respectively.*

Notice that the characteristic bundle is a generalization of the characteristic function of a quasi-toric manifold, or a torus manifold.

Example 3.4. For a $2n$-dimensional manifold X equipped with a locally standard T^n-action, $\pi_{P_X} : P_X \to B_X$ is the trivial principal $\mathrm{Aut}(T^n)$-bundle $P_X = B_X \times \mathrm{Aut}(T^n)$. Let $(\mathcal{S}^{(n-1)}B_X)_a$ $(a = 1, \ldots, k)$ be the connected component of $\mathcal{S}^{(n-1)}B_X$. On the preimage $\mu_X^{-1}((\mathcal{S}^{(n-1)}B_X)_a)$ of each connected component

$(\mathcal{S}^{(n-1)}B_X)_a$ by μ_X, T^n-action on it has the unique one-dimensional stabilizer which we denote by S^1_a. Let $\mathcal{L}_a$ be the rank one sub-lattice in Λ corresponding to S^1_a. Then, $\mathcal{L}_X$ is the disjoint union $\coprod_a (\mathcal{S}^{(n-1)}B_X)_a \times \mathcal{L}_a$.

Example 3.5. In the case of Example 2.8, the characteristic pair is constructed as follows. We identify Λ with $\mathbb{Z}^2$ and also identify $\mathrm{Aut}(T^2)$ with $\mathrm{GL}_2(\mathbb{Z})$. Then P_X can be written by

$$P_X = ((0, 1+\varepsilon) \times [0,1] \times \mathrm{GL}_2(\mathbb{Z})) \,/ \sim_P,$$

where $(\xi, A) \sim_P (\eta, B)$ if and only if $\eta \sim_B \xi$ and $B = -A$. The bundle projection is defined by the obvious way. Λ_X is written by the similar way, namely,

$$\Lambda_X = ((0, 1+\varepsilon) \times [0,1] \times \mathbb{Z}^2) \,/ \sim_\Lambda,$$

where $(\xi, m) \sim_P (\eta, n)$ if and only if $\eta \sim_B \xi$ and $n = -m$. With this notation, $\mathcal{L}_X$ is written by

$$\mathcal{L}_X = ((0, 1+\varepsilon) \times \{0,1\} \times \{0\} \oplus \mathbb{Z}) \,/ \sim_\Lambda .$$

For $i = 1, 2$, let B_i be an n-dimensional topological manifold with corners and $(P_i, \mathcal{L}_i)$ a pair of a principal $\mathrm{Aut}(T^n)$-bundle $\pi_{P_i} : P_i \to B_i$ and a unimodular rank one sub-bundle $\pi_{\mathcal{L}_i} \colon \mathcal{L}_i \to \mathcal{S}^{(n-1)}B_i$ of the restriction of the associated Λ-bundle $\pi_{\Lambda_i} \colon \Lambda_{P_i} \to B_i$ of P_i by the natural identification $\mathrm{Aut}(T^n) \cong \mathrm{GL}(\Lambda)$ to the codimension one stratum $\mathcal{S}^{(n-1)}B_i$ of B_i.

Definition 3.6. An *isomorphism* $f_P \colon (P_1, \mathcal{L}_1) \to (P_2, \mathcal{L}_2)$ from $(P_1, \mathcal{L}_1)$ to $(P_2, \mathcal{L}_2)$ is a bundle isomorphism $f_P \colon P_1 \to P_2$ which covers a stratification preserving homeomorphism $f_B \colon B_1 \to B_2$ such that the lattice bundle isomorphism $f_\Lambda : \Lambda_{P_1} \to \Lambda_{P_2}$ induced by f_P sends $\mathcal{L}_1$ isomorphically to $\mathcal{L}_2$. $(P_1, \mathcal{L}_1)$ and $(P_2, \mathcal{L}_2)$ are *isomorphic* if there exists an isomorphism between them.

The isomorphism class of the characteristic pair $(P_X, \mathcal{L}_X)$ is an invariant of a local T^n-action on X.

Lemma 3.7. *For $i = 1, 2$, let $(X_i, \mathcal{T}_i)$ be a $2n$-dimensional manifold X_i with a local T^n-action $\mathcal{T}_i$. If there is a C^0 isomorphism $f_X \colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$, then f_X induces the isomorphism $f_{P_X} \colon (P_{X_1}, \mathcal{L}_{X_1}) \to (P_{X_2}, \mathcal{L}_{X_2})$ between characteristic pairs associated with X_1 and X_2.*

Proof. Let $\{(U^{X_1}_\beta, \varphi^{X_1}_\beta)\}_{\beta \in \mathcal{B}} \in \mathcal{T}_1$ and $\{(U^{X_2}_\alpha, \varphi^{X_2}_\alpha)\}_{\alpha \in \mathcal{A}} \in \mathcal{T}_2$ be maximal weakly standard atlases of X_1 and X_2, and $\{(U^{B_1}_\beta, \varphi^{B_1}_\beta)\}_{\beta \in \mathcal{B}}$ and $\{(U^{B_2}_\alpha, \varphi^{B_2}_\alpha)\}_{\alpha \in \mathcal{A}}$ atlases of B_{X_1} and B_{X_2} induced by $\{(U^{X_1}_\beta, \varphi^{X_1}_\beta)\}_{\beta \in \mathcal{B}}$ and $\{(U^{X_2}_\alpha, \varphi^{X_2}_\alpha)\}_{\alpha \in \mathcal{A}}$, respectively. Suppose that $f_X \colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is a C^0 isomorphism and f_B is the homeomorphism from B_{X_1} to B_{X_2} which is induced by f_X. By definition, on each nonempty overlap $U^{B_1}_\beta \cap f_B^{-1}(U^{B_2}_\alpha)$, there exists an automorphism $\rho^f_{\alpha\beta}$ of T^n such that $\varphi^{X_2}_\alpha \circ f_X \circ (\varphi^{X_1}_\beta)^{-1}$ is $\rho^f_{\alpha\beta}$-equivariant. It is easy to see that the following equality holds

$$\rho^f_{\alpha_0,\beta_0} \circ \rho^{X_1}_{\beta_0\beta_1} = \rho^{X_2}_{\alpha_0\alpha_1} \circ \rho^f_{\alpha_1\beta_1} \tag{3.3}$$

on a nonempty intersection $U^{B_1}_{\beta_0\beta_1} \cap f_B^{-1}(U^{B_2}_{\alpha_0\alpha_1})$, where $\rho^{X_1}_{\beta_0\beta_1}$ and $\rho^{X_2}_{\alpha_0\alpha_1}$ are automorphisms of T^n in (2) of Definition 2.1 with respect to X_1 and X_2, respectively. We define the bundle isomorphism $(f_P)_{\alpha\beta} \colon U^{B_1}_\beta \cap f_B^{-1}(U^{B_2}_\alpha) \times \mathrm{Aut}(T^n) \to f_B(U^{B_1}_\beta) \cap U^{B_2}_\alpha \times \mathrm{Aut}(T^n)$ by

$$(f_P)_{\alpha\beta}(b, h) := (f_B(b), \rho^f_{\alpha\beta} \circ h).$$

By (3.3), we can patch them together to obtain the bundle isomorphism $f_P \colon P_{X_1} \to P_{X_2}$ which covers f_B. $\square$

3.2. **Canonical models.** In [4, Section 1.5], Davis-Januszkiewicz constructed the canonical model of a quasi-toric manifold from the based polytope and the characteristic function. A similar construction can be done by using the characteristic pair in the following way. Let B be an n-dimensional C^0 manifold with corners and $(P, \mathcal{L})$ a characteristic pair on B. We denote by $\pi_T : T_P \to B$ the T^n-bundle associated with P by the natural action of $\mathrm{Aut}(T^n)$ on T^n. First we shall explain that for any k-dimensional part $\mathcal{S}^{(k)}B$, $(P, \mathcal{L})$ determines a rank $n-k$ sub-torus bundle of the restriction of $\pi_T : T_P \to B$ to $\mathcal{S}^{(k)}B$. Let $\{U_\alpha\}$ be an open covering of B such that on each U_α there exists a local trivialization $\varphi_\alpha^P : \pi_P^{-1}(U_\alpha) \to U_\alpha \times \mathrm{Aut}(T^n)$. On each nonempty overlap $U_{\alpha\beta}$ we denote by $\rho_{\alpha\beta}$ the transition function with respect to φ_α^P and φ_β^P, namely,

$$\varphi_\alpha^P \circ (\varphi_\beta^P)^{-1}(b, f) = (b, \rho_{\alpha\beta} f)$$

for $(b, f) \in U_\beta \times \mathrm{Aut}(T^n)$. Notice that $\rho_{\alpha\beta}$ is locally constant since $\mathrm{Aut}(T^n)$ is discrete. φ_α^P induces the local trivializations of the associated bundles T_P and Λ_P which are denoted by $\varphi_\alpha^T : \pi_T^{-1}(U_\alpha) \to U_\alpha \times T^n$ and $\varphi_\alpha^\Lambda : \pi_\Lambda^{-1}(U_\alpha) \to U_\alpha \times \Lambda$, respectively. For $\mathcal{S}^{(k)}B$ we take U_α with $U_\alpha \cap \mathcal{S}^{(k)}B \neq \emptyset$. By replacing U_α by a sufficiently small one if necessary, we may assume that the intersection $U_\alpha \cap \mathcal{S}^{(n-1)}B$ of U_α with the codimension one part $\mathcal{S}^{(n-1)}B$ of B has exactly $n-k$ connected components, say $(U_\alpha \cap \mathcal{S}^{(n-1)}B)_1, \cdots, (U_\alpha \cap \mathcal{S}^{(n-1)}B)_{n-k}$. For $k = n$, this means that U_α is contained in $\mathcal{S}^{(n)}B$. For $k < n$, there are $n-k$ rank one sub-lattices $L_1, \ldots, L_{n-k}$ of Λ such that for $a = 1, \ldots, n-k$ φ_α^Λ sends the restriction of $\pi_{\mathcal{L}} \colon \mathcal{L} \to \mathcal{S}^{(n-1)}B$ to $(U_\alpha \cap \mathcal{S}^{(n-1)}B)_a$ isomorphically to the trivial rank one subbundle $(U_\alpha \cap \mathcal{S}^{(n-1)}B)_a \times L_a$ of $(U_\alpha \cap \mathcal{S}^{(n-1)}B)_a \times \Lambda$. Since $\mathcal{L}$ is unimodular, $L_1, \ldots, L_{n-k}$ generate the $(n-k)$-dimensional sub-torus of T^n which is denoted by $Z_{U_\alpha \cap \mathcal{S}^{(k)}B}$. For $k = n$, we define $Z_{U_\alpha \cap \mathcal{S}^{(n)}B}$ to be the trivial subgroup which consists of the unit element. Notice that when $(P, \mathcal{L})$, $\{U_\alpha\}$, and φ_α^P are the ones induced by some local T^n-action $\mathcal{T}$ on X, $Z_{U_\alpha^B \cap \mathcal{S}^{(k)}B_X}$ is the common $(n-k)$-dimensional stabilizer of T^n-action on $\mu_{\mathbb{C}^n}^{-1}(U_\alpha^B \cap \mathcal{S}^{(k)}B_X)$.

Suppose that another U_β satisfies the above condition and $U_{\alpha\beta} \cap \mathcal{S}^{(k)}B \neq \emptyset$. By the definition of $(P, \mathcal{L})$, $\rho_{\alpha\beta}$ sends $Z_{U_\beta^B \cap \mathcal{S}^{(k)}B_X}$ isomorphically to $Z_{U_\alpha^B \cap \mathcal{S}^{(k)}B_X}$. Hence, in the same way as before, they are patched together to form a rank $n-k$ sub-torus bundle, which is denoted by $\pi_{Z_{\mathcal{S}^{(k)}B}} : Z_{\mathcal{S}^{(k)}B} \to \mathcal{S}^{(k)}B$, of the restriction of $\pi_T \colon T_P \to B$ to $\mathcal{S}^{(k)}B$.

Definition 3.8. For $t, t' \in T_P$, t and t' are *equivalent* or $t \sim_{can} t'$ if and only if $\pi_T(t) = \pi_T(t')$ and $t't^{-1} \in \pi_{Z_{\mathcal{S}^{(k)}B}}^{-1}(\pi_T(t))$ when $\pi_T(t)$ lies in $\mathcal{S}^{(k)}B$. Notice that a fiber of $\pi_T \colon T_P \to B$ is equipped with the structure of a group since its structure group is $\mathrm{Aut}(T^n)$.

We denote by $X_{(P,\mathcal{L})}$ the quotient space of T_P by the equivalence relation. The bundle projection $\pi_T \colon T_P \to B$ descends to the map $\mu_{X_{(P,\mathcal{L})}} \colon X_{(P,\mathcal{L})} \to B$. On any U_α, under the identification $\varphi_\alpha^T \colon \pi_T^{-1}(U_\alpha) \to U_\alpha \times T^n$, the equivalence relation in Definition 3.8 can be rewritten as follows. For (b, t), $(b', t') \in U_\alpha \times T^n$, $(b, t) \sim_{can} (b', t')$ if and only if $b = b'$ and $t't^{-1} \in Z_{U_\alpha \cap \mathcal{S}^{(k)}B}$ when b lies in $\mathcal{S}^{(k)}B$. Then, φ_α^T induces the identification of $\mu_{X_{(P,\mathcal{L})}}^{-1}(U_\alpha)$ with $(U_\alpha \times T^n)/\sim_{can}$ on U_α. Now we take $\{U_\alpha\}$ to be an atlas $\{(U_\alpha, \varphi_\alpha^B)\}$ of B as a manifold with corners. Since $\mathcal{L}$ is unimodular and B is a manifold with corners, by the same way as in Davis-Januszkiewicz [4, Section 1.5], or Masuda-Panov [8, Section 3.2], we can show that $(U_\alpha \times T^n)/\sim_{can}$ is also homeomorphic to a T^n-invariant open subset $\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha))$ of $\mathbb{C}^n$. Hence, by taking the composition of these identifications, there is a homeomorphism $\varphi_\alpha^{X_{(P_X, \mathcal{L}_X)}} \colon \mu_{X_{(P_X, \mathcal{L}_X)}}^{-1}(U_\alpha) \to \mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha))$ which covers

$\varphi_\alpha^B\colon U_\alpha \to \varphi_\alpha^B(U_\alpha)$. Notice that on $U_{\alpha\beta}$ the overlap map with these identifications is induced by $\mathrm{id}_{U_{\alpha\beta}} \times \rho_{\alpha\beta}\colon U_{\alpha\beta}\times T^n \to U_{\alpha\beta}\times T^n$. Hence, $X_{(P,\mathcal{L})}$ is a $2n$-dimensional topological manifold equipped with a C^0 local T^n-action whose orbit space is B and whose orbit map is $\mu_{X_{(P,\mathcal{L})}}$.

Definition 3.9. We call $X_{(P,\mathcal{L})}$ the *canonical model* of $(P,\mathcal{L})$. In particular, when $(P,\mathcal{L})$ is the characteristic pair $(P_X,\mathcal{L}_X)$ of a local T^n-action $\mathcal{T}$ on a $2n$-dimensional manifold X, we also call $X_{(P_X,\mathcal{L}_X)}$ the canonical model of $(X,\mathcal{T})$.

The following propositions describe the properties of the canonical model. For proofs see [13].

Proposition 3.10. *For any characteristic pair $(P,\mathcal{L})$, $\mu_{X_{(P,\mathcal{L})}}\colon X_{(P,\mathcal{L})} \to B$ admits a continuous section s.*

For any characteristic pair $(P,\mathcal{L})$, recall that a fiber of T_P admits a structure of a group. By the construction, a fiber of $\mu_{X_{(P,\mathcal{L})}}\colon X_{(P,\mathcal{L})} \to B$ also admits a group structure.

Proposition 3.11 ([13]). *For a $2n$-dimensional manifold $(X,\mathcal{T})$ equipped with a local T^n-action, we denote the associated T^n-bundle T_{P_X} of P_X by $\pi_{T_X}\colon T_X \to B_X$ for simplicity. Then T_X acts fiber-wise on X. Similarly $X_{(P_X,\mathcal{L}_X)}$ also acts fiber-wise on X. For any $b \in B_X$ the action of $\mu^{-1}_{X_{(P,\mathcal{L})}}(b)$ on $\mu_X^{-1}(b)$ is simply transitive.*

The following lemma follow directly from the construction of a canonical model.

Lemma 3.12. *For $i = 1, 2$, let B_i be an n-dimensional topological manifold with corners and $(P_i,\mathcal{L}_i)$ a characteristic pair on B_i. Then, any isomorphism $f_P\colon (P_1,\mathcal{L}_1) \to (P_2,\mathcal{L}_2)$ induces the C^0 isomorphism $f_{X_{(P,\mathcal{L})}}\colon X_{(P_1,\mathcal{L}_1)} \to X_{(P_2,\mathcal{L}_2)}$ between canonical models of $(P_1,\mathcal{L}_1)$ and $(P_2,\mathcal{L}_2)$.*

Remark 3.13. If there is an isomorphism $f_P\colon (P_1,\mathcal{L}_1) \to (P_2,\mathcal{L}_2)$ between characteristic pairs, then the induced C^0 isomorphism $f_{X_{(P,\mathcal{L})}}\colon X_{(P_1,\mathcal{L}_1)} \to X_{(P_2,\mathcal{L}_2)}$ between canonical models is fiber-wise group isomorphism.

4. The Euler classes of orbit maps

In this section, for a local torus action we define the Euler class of the orbit map as an obstruction class for the orbit map to have a continuous section. In this section we assume that manifolds, maps, and local T^n-actions are of class C^0 unless otherwise stated. Let $(X,\mathcal{T})$ be a $2n$-dimensional manifold equipped with a local T^n-action. We investigate when $\mu_X\colon X \to B_X$ has a section. We assume that the index set $\mathcal{A}$ of the weakly standard atlas $\{(U_\alpha^X,\varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$ is countable ordered. By the construction of $X_{(P_X,\mathcal{L}_X)}$, there exists a C^0 isomorphism $h_\alpha\colon \mu_X^{-1}(U_\alpha^B) \to \mu^{-1}_{X_{(P_X,\mathcal{L}_X)}}(U_\alpha^B)$ covering the identity on each U_α^B such that h_α is equivariant with respect to the fiber-wise action of T_X or $X_{(P_X,\mathcal{L}_X)}$. (For example we can take $(\varphi_\alpha^{X_{(P_X,\mathcal{L}_X)}})^{-1}\circ\varphi_\alpha^X$ as h_α.) On each nonempty overlap $U_{\alpha\beta}^B$ the equation

$$h_\alpha \circ h_\beta^{-1}(x) = \theta_{\alpha\beta}^X(b)x \tag{4.1}$$

for $b \in U_{\alpha\beta}^B$ and $x \in \mu^{-1}_{X_{(P_X,\mathcal{L}_X)}}(b)$ determines a unique local section $\theta_{\alpha\beta}^X$ of $\mu_{X_{(P_X,\mathcal{L}_X)}}$ on $U_{\alpha\beta}^B$. Let $\mathscr{S}_{(P_X,\mathcal{L}_X)}$ denote the sheaf of germs of continuous sections of $\mu_{X_{(P_X,\mathcal{L}_X)}}$. Then local sections $\theta_{\alpha\beta}^X$ form a Čech one-chain $\{\theta_{\alpha\beta}^X\}$ on $\{U_\alpha^B\}$ with values in $\mathscr{S}_{(P_X,\mathcal{L}_X)}$. Moreover, by definition, we can show the following lemma.

Lemma 4.1. *$\{\theta_{\alpha\beta}^X\}$ is a cocycle.*

Let $H^1(B_X;\mathscr{S}_{(P_X,\mathcal{L}_X)})$ denote the first Čech cohomology group of B_X with values in $\mathscr{S}_{(P_X,\mathcal{L}_X)}$. By the above lemma, $\{\theta^X_{\alpha\beta}\}$ defines the cohomology class in $H^1(B_X;\mathscr{S}_{(P_X,\mathcal{L}_X)})$. We denote it by $e_{orbit}(X)$. It is easy to see that $e_{orbit}(X)$ does not depend on the choice of h_αs and depends only on the local T^n-action on X.

Definition 4.2. We call $e_{orbit}(X)$ the *Euler class of* μ_X.

Notice that if the local T^n-action is induced by a locally standard T^n-action and $\partial B_X = \emptyset$, then $\mu_X\colon X \to B_X$ is a principal T^n-bundle. In this case, $e_{orbit}(X)$ is nothing but the Euler class of the principal T^n-bundle.

Theorem 4.3. *$\mu_X\colon X \to B_X$ has a section if and only if $e_{orbit}(X)$ vanishes.*

Example 4.4. For the T^n-action on a complex n-dimensional, nonsingular toric variety X, $e_{orbit}(X)$ vanishes.

Example 4.5. For Example 2.8, $e_{orbit}(X)$ vanishes. In fact, we can defined the section s of $\mu_X\colon X \to B_X$ by

$$s([\xi_1,\xi_2]) := [(\sqrt{\xi_1},\sqrt{\xi_2}),\sqrt{1-\xi_2}]$$

for $[\xi_1,\xi_2] \in B_X$.

For $i = 1,2$, let B_i be an n-dimensional topological manifold with corners and $(P_i,\mathcal{L}_i)$ a characteristic pair on B_i. Suppose that there exists an isomorphism $f_P\colon (P_1,\mathcal{L}_1) \to (P_2,\mathcal{L}_2)$. By Lemma 3.12, it induces the isomorphism $f_P^*\colon H^1(B_2;\mathscr{S}_{(P_2,\mathcal{L}_2)}) \to H^1(B_1;\mathscr{S}_{(P_1,\mathcal{L}_1)})$ between cohomology groups. In particular, by Lemma 3.7 and Lemma 3.12, a C^0 isomorphism $f_X\colon (X_1,\mathcal{T}_1) \to (X_2,\mathcal{T}_2)$ induces the isomorphism $f_{P_X}^*\colon H^1(B_{X_2};\mathscr{S}_{(P_{X_2},\mathcal{L}_{X_2})}) \to H^1(B_{X_1};\mathscr{S}_{(P_{X_1},\mathcal{L}_{X_1})})$.

Lemma 4.6. *For $i = 1,2$, let $(X_i,\mathcal{T}_i)$ be a $2n$-dimensional manifold equipped with a local T^n-action. If there is a C^0 isomorphism $f_X\colon X_1 \to X_2$, then $f_{P_X}^* e_{orbit}(X_2) = e_{orbit}(X_1)$.*

5. The topological classification

The following is the main theorem of [13].

Theorem 5.1 ([13]). *For $i = 1,2$, let $(X_i,\mathcal{T}_i)$ be a $2n$-dimensional manifold X_i with a local T^n-action $\mathcal{T}_i$. X_1 and X_2 are C^0 isomorphic if and only if there exists an isomorphism $f_P\colon (P_{X_1},\mathcal{L}_{X_1}) \to (P_{X_2},\mathcal{L}_{X_2})$ between characteristic pairs associated with X_1 and X_2 such that $f_P^* e_{orbit}(X_2) = e_{orbit}(X_1)$. Moreover, for any characteristic pair $(P,\mathcal{L})$ on an n-dimensional topological manifold B with corners and for any element $e \in H^1(B;\mathscr{S}_{(P,\mathcal{L})})$, there exists a $2n$-dimensional C^0 manifold $(X,\mathcal{T})$ equipped with a C^0 local T^n-action whose characteristic pair and the Euler class of the orbit map are equal to $(P,\mathcal{L})$ and e, respectively.*

The idea of the proof. The only if part follows from Lemma 3.7 and Lemma 4.6. The proof of the if part is similar to the proof of the classification of principal bundles and the idea is as follows. Recall that by definition, $e_{orbit}(X)$ measures the difference between X and $X_{(P_X,\mathcal{L}_X)}$. If there is an isomorphism $f_P\colon (P_{X_1},\mathcal{L}_{X_1}) \to (P_{X_2},\mathcal{L}_{X_2})$, then, by Lemma 3.12, f_P induces the C^0 isomorphism from $X_{(P_{X_1},\mathcal{L}_{X_1})}$ to $X_{(P_{X_2},\mathcal{L}_{X_2})}$. Moreover, suppose that $f_P^* e_{orbit}(X_2) = e_{orbit}(X_1)$. This means that the difference between X_1 and $X_{(P_{X_1},\mathcal{L}_{X_1})}$ is same as the difference between X_2 and $X_{(P_{X_2},\mathcal{L}_{X_2})}$ under the identification $X_{(P_{X_1},\mathcal{L}_{X_1})} \cong X_{(P_{X_2},\mathcal{L}_{X_2})}$. Hence, X_1 is C^0 isomorphic to X_2. For more details, see [13]. $\square$

We focus on the case of locally standard torus actions. We remark that if a manifold X is equipped with a locally standard torus action, then, P_X is the trivial bundle $P_X = B_X \times \mathrm{Aut}(T^n)$. In this case, we can obtain the following corollary. It is a generalization of the topological classification theorem for effective T^2-actions on four-dimensional manifolds without finite stabilizers by Orlik-Raymond [10] and for quasi-toric manifolds by Davis-Januszkiewicz [4].

Corollary 5.2 ([13]). *Locally standard torus actions are classified by the characteristic bundle and the Euler class of the orbit map up to equivariant homeomorphisms.*

6. Locally toric Lagrangian fibrations

Let $(X, \mathcal{T})$ be a $2n$-dimensional smooth manifold equipped with a smooth local T^n-action $\mathcal{T}$. In this section, we investigate the condition in order that $\mu_X \colon X \to B_X$ becomes a locally toric Lagrangian fibration.

Lemma 6.1. *Suppose that there exists a symplectic structure ω on X and there also exists a weakly standard atlas $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$ of X such that on each U_α^X, φ_α^X preserves symplectic forms, namely, $\omega = {\varphi_\alpha^X}^* \omega_{\mathbb{C}^n}$. For each nonempty overlap $U_{\alpha\beta}^X \neq \emptyset$, let $\rho_{\alpha\beta} \in \mathrm{Aut}(T^n)$ be the automorphism in* (2) *of Definition* 2.1 *with respect to $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$. We identify $\rho_{\alpha\beta}$ with an element of $\mathrm{GL}_n(\mathbb{Z})$ by the natural identification $\mathrm{Aut}(T^n) \cong \mathrm{GL}_n(\mathbb{Z})$. Let $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ be the atlas of B_X induced by $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$. Then, on each nonempty overlap $U_{\alpha\beta}^B \neq \emptyset$, the overlap map $\varphi_{\alpha\beta}^B \colon \varphi_\beta^B(U_{\alpha\beta}^B) \to \varphi_\alpha^B(U_{\alpha\beta}^B)$ is of the form*

$$\varphi_{\alpha\beta}^B(\xi) = \rho_{\alpha\beta}^{-T}(\xi) + c_{\alpha\beta}, \tag{6.1}$$

for some constant $c_{\alpha\beta}$, where $\rho_{\alpha\beta}^{-T}$ is the transpose inverse of $\rho_{\alpha\beta}$. In particular, B_X becomes a smooth manifold with corners.

Proof. Let $\omega_{\mathbb{R}^n\times T^n}$ be the symplectic form on $\mathbb{R}^n \times T^n$ which is defined by

$$\omega_{\mathbb{R}^n\times T^n} = \sum_{k=1}^n d\theta_k \wedge d\xi_k,$$

where $(\xi_1, \dots, \xi_n)$ is the standard coordinates of $\mathbb{R}^n$ and $(\theta_1, \dots, \theta_n)$ is the angle coordinates of T^n with period 1, which means $(e^{2\pi\theta_1}, \dots, e^{2\pi\theta_n}) \in T^n$. First we focus on the interior of B_X. We can show that for each α, there exists a symplectomorphism $\phi_\alpha \colon (\mu_X^{-1}(U_\alpha^B \setminus \partial B_X), \omega) \to (\varphi_\alpha^B(U_\alpha^B \setminus \partial B_X) \times T^n, \omega_{\mathbb{R}^n\times T^n})$ such that $\mathrm{pr}_1 \circ \phi_\alpha = \varphi_\alpha^B \circ \mu_X$ and on an overlap $U_{\alpha\beta}^B$, the overlap map $\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1}$ is of the form $\phi_{\alpha\beta}(b, u) = (\varphi_{\alpha\beta}^B(b), \rho_{\alpha\beta}(u) u_{\alpha\beta}(b))$ for some map $u_{\alpha\beta} \colon U_{\alpha\beta}^B \to T^n$, where $\mathrm{pr}_1 \colon \varphi_\alpha^B(U_\alpha^B \setminus \partial B_X) \times T^n \to \varphi_\alpha^B(U_\alpha^B \setminus \partial B_X)$ is the natural projection to the first factor. For more details, see [13]. Then, by [11, Lemma 2.5], on each overlap $U_{\alpha\beta}^B \setminus \partial B_X$ the overlap map $\varphi_{\alpha\beta}^B$ is of the form (6.1). Since $U_{\alpha\beta}^B \setminus \partial B_X$ is open dense in $U_{\alpha\beta}^B$, $\varphi_{\alpha\beta}^B$ should be of the form (6.1) on the whole $U_{\alpha\beta}^B$. □

Definition 6.2. We call the atlas $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ of B_X in Lemma 6.1 an *integral affine structure* compatible with $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}}$.

Let $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$ be a weakly standard atlas of X. Suppose that the induced atlas $\{(U_\alpha^B, \varphi_\alpha^B)\}_{\alpha\in\mathcal{A}}$ of B_X is an integral affine structure compatible with $\{(U_\alpha^X, \varphi_\alpha^X)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$.

Lemma 6.3. *The characteristic bundle $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to \mathcal{S}^{(n-1)}B_X$ admits a smooth section which generates $\mathcal{L}_X$ fiber-wisely. In particular, $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to \mathcal{S}^{(n-1)}B_X$ is determined by the integral affine structure.*

Proof. Let $(U^B_\alpha, \varphi^B_\alpha)$ be a coordinate neighborhood of B_X with $U^B_\alpha \cap \mathcal{S}^{(n-1)}B_X \neq \emptyset$. We may assume that the intersection $U^B_\alpha \cap \mathcal{S}^{(n-1)}B_X$ is connected. (Otherwise, we may consider component-wise.) As described in the construction of $\mathcal{L}_X$, the local trivialization φ^Λ_α of Λ_X sends $\pi^{-1}_{\mathcal{L}_X}(U^B_\alpha \cap \mathcal{S}^{(n-1)}B_X)$ isomorphically to $U^B_\alpha \cap \mathcal{S}^{(n-1)}B_X \times \mathcal{L}_\alpha$, where $\mathcal{L}_\alpha$ is a rank one sublattice of Λ. Then there exists a unique generator u_α of $\mathcal{L}_\alpha$ such that $\varphi^B_\alpha(U^B_\alpha)$ and $\varphi^B_\alpha(U^B_\alpha \cap \mathcal{S}^{(n-1)}B_X)$ lie in the upper half space $\{\xi \in \mathbb{R}^n \colon \langle \xi, u_\alpha\rangle \geq 0\}$ and the hyperplane $\{\xi \in \mathbb{R}^n \colon \langle \xi, u_\alpha\rangle = 0\}$ determined by u_α, respectively. Suppose that $(U^B_\beta, \varphi^B_\beta)$ is another coordinate neighborhoods satisfying the above conditions and the intersection $U^B_{\alpha\beta} \cap \mathcal{S}^{(n-1)}B_X$ is nonempty. Let u_β be the corresponding generator of $\mathcal{L}_\beta$. Since the overlap map $\varphi^B_{\alpha\beta}$ is of the form (6.1), $\varphi^B_{\alpha\beta}$ sends $\{\xi \in \mathbb{R}^n \colon \langle \xi, u_\beta\rangle \geq 0\}$ and $\{\xi \in \mathbb{R}^n \colon \langle \xi, u_\beta\rangle = 0\}$ diffeomorphically to $\{\xi \in \mathbb{R}^n \colon \langle \xi, u_\alpha\rangle \geq 0\}$ and $\{\xi \in \mathbb{R}^n \colon \langle \xi, u_\alpha\rangle = 0\}$, respectively. In particular, this implies that $u_\alpha = \rho_{\alpha\beta}(u_\beta)$. Thus u_α's form the required section of $\mathcal{L}_X$. □

By (6.1) the structure group of the cotangent bundle T^*B_X is $\mathrm{GL}_n(\mathbb{Z})$ and the principal $\mathrm{Aut}(T^n)$-bundle P_X is nothing but the frame bundle of T^*B_X. Now we have the following exact sequence of associated fiber bundles of P_X

$$0 \longrightarrow \Lambda_X \longrightarrow T^*B_X \longrightarrow T_X \longrightarrow 0.$$

As is well-known, T^*B_X is equipped with the standard symplectic structure, and it is easy to see that the standard symplectic structure on T^*B_X descends to the symplectic structure on T_X, which is denoted by ω_{T_X}, so that $\pi_{T_X} \colon (T_X, \omega_{T_X}) \to B_X$ is a nonsingular Lagrangian fibration. Moreover, we can show that following lemma.

Lemma 6.4. *The canonical model $X_{(P_X, \mathcal{L}_X)}$ becomes a smooth locally toric Lagrangian fibration on B_X.*

Roughly speaking, the proof is as follows. For each U^B_α, the section of $\mathcal{L}_X$ defines a Hamiltonian action of some sub-torus of T^n on $\pi^{-1}_{T_X}(U^B_\alpha)$. $X_{(P_X,\mathcal{L}_X)}$ can be obtained by symplectic cutting technique with respect to these Hamiltonian torus actions. For more details, see [13].

From Lemma 6.4, in particular, $h_\alpha \colon \mu^{-1}_X(U^B_\alpha) \to \mu^{-1}_{X_{(P_X,\mathcal{L}_X)}}(U^B_\alpha)$ in Section 4 can be taken to be a C^∞ isomorphism which covers the identity on each U^B_α and $\theta^X_{\alpha\beta}$ defined by (4.1) can be also taken to be a C^∞ local section of T_X on $U^B_{\alpha\beta}$. Then the necessary and sufficient condition in order that $\mu_X \colon X \to B_X$ becomes a locally toric Lagrangian fibration is given as follows.

Lemma 6.5. *Let $(X, \mathcal{T})$ be a $2n$-dimensional smooth manifold equipped with a smooth local T^n-action $\mathcal{T}$. There exists a symplectic structure ω on X and there also exists a weakly standard atlas $\{(U^X_\alpha, \varphi^X_\alpha)\}_{\alpha\in\mathcal{A}} \in \mathcal{T}$ of X such that on each U^X_α, $\omega = \varphi^{X*}_\alpha \omega_{\mathbb{C}^n}$ if and only if the atlas $\{(U^B_\alpha, \varphi^B_\alpha)\}_{\alpha\in\mathcal{A}}$ of B_X induced by $\{(U^X_\alpha, \varphi^X_\alpha)\}_{\alpha\in\mathcal{A}}$ is an integral affine structure compatible with $\{(U^X_\alpha, \varphi^X_\alpha)\}_{\alpha\in\mathcal{A}}$ and on each nonempty overlap $U^B_{\alpha\beta}$, $\theta^X_{\alpha\beta}$ is a Lagrangian section, namely, $(\theta^X_{\alpha\beta})^*\omega_{T_X}$ vanishes.*

For nonsingular Lagrangian fibrations, this result is obtained by Duistermaat [5]. See also [11], [9]. Recently, in [6] Gay-Symington showed the similar result for near-symplectic four-manifolds.

Finally we state the classification theorem for locally toric Lagrangian fibrations. For a locally toric Lagrangian fibration $\mu \colon (X, \omega) \to B$, the local sections $\theta^X_{\alpha\beta}$ define a Čech cohomology class $\lambda(X) \in H^1(B_X; \mathcal{S}^{Lag}_{T_X})$ of B_X with values in the sheaf $\mathcal{S}^{Lag}_{T_X}$ of germs of Lagrangian sections of $\pi_{T_X} \colon (T_X, \omega_{T_X}) \to B_X$.

Theorem 6.6 ([1], [13]). *Locally toric Lagrangian fibrations are classified by integral affine structures on the bases and $\lambda(X)$ up to fiber-preserving symplectomorphisms.*

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